One-dimensional contact process: Duality and renormalization

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We study the one-dimensional contact process in its quantum version using a recently proposed real-space renormalization technique for stochastic many-particle systems. Exploiting the duality and other properties of the model, we can apply the method for cells with up to 37 sites. After suitable extrapolation, we obtain exponent estimates that are comparable in accuracy with the best known in the literature.

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I. INTRODUCTION

Phase transitions out of an absorbing state form an important class of nonequilibrium critical phenomena [1]. Models having such a transition have appeared in various areas such as surface chemistry [2], population dynamics [3], etc. Very recently, it was even shown that the so-called ''self-organized criticality'' appearing in a number of sandpile models can be related to ''ordinary'' criticality in a class of models with an infinite number of absorbing states and a conservation law [4,5].

A hot topic in current nonequilibrium statistical mechanics is to understand what the possible universality classes are that can exist for such models, and more importantly, what precisely determines these universality classes. Research on these questions has mainly focused on the one-dimensional case. By now, it is clear that two main nontrivial universality classes exist. The first one is that of directed percolation (DP), which contains models such as the contact process, the Ziff-Gulari-Barshad [2] model of catalysis, branching and annihilating walks with an odd offspring [6], etc. This class is very robust, in agreement with *the DP conjecture* [7], which states that all phase transitions out of an absorbing state in models with a scalar order parameter, short range interactions, and no conservation laws belong to the DP universality class. Over the past decade, the existence of a second universality class has been clearly established. This class, known as the parity-conserving (PC) class, contains, among others, such models as the branching and annihilating walks with an even offspring, the monomer-dimer model [8], and a certain type of generalized contact process [9]. The precise conditions that determine this class are, however, still unclear. Some authors argue that it is a conservation law [10] (the conservation of particle number modulo 2) that is the important factor, whereas others have claimed that it is the existence of two equivalent absorbing states [9].

Another issue that is currently being debated is the importance of exclusion in these models. Indeed, it has been recently argued that adding exclusion to a model of branching and annihilating walks with $N \geqslant 2$ species of particles changes some of the critical exponents [11].

In light of these questions, the development of precise approximate techniques is crucial. Most of the current understanding of these models has come from two approaches: extensive numerical simulations and field theoretic renormalization-group (RG) techniques [12,13]. Both methods have their strong and weak points. Simulations allow the study of quite big system sizes (especially in $d=1$). Near the critical point, however, relaxation times may be quite large and one can never be very sure that the asymptotic time regime has been reached. Field theoretic techniques are very powerful but have their own difficulties. In the case of the branching and annihilating random walks with an even offspring, there exist two upper critical dimensions that make reliable exponent estimates in $d=1$ very difficult [10]. Besides these two main techniques, also a series expansion method has been developed for nonequilibrium systems [14,15]. This method gives very accurate exponent estimates for such models as the contact process, but has not been applied very extensively yet.

For these reasons, in recent years attention has been given to alternative approaches that can be called real-space renormalization methods. These techniques use the by now well known formal equivalence between a stochastic system and a quantum-mechanical model evolving in imaginary time [16]. In this way, one can associate with the generator of a Markov chain a quantum Hamiltonian, which can then be studied using various techniques that were originally introduced in the study of quantum spin chains, fermion models, etc. For some models, this approach can lead to an exact solution. As an example, we mention the relation between the asymmetric exclusion process and the XXZ chain. Unfortunately, no models with a nontrivial bulk phase transition can be solved in this way. Yet, in the same spirit one can then use approximate techniques originally developed for quantum systems to study stochastic systems. The most famous of these approaches is certainly the density-matrix renormalization group (DMRG) [17,18]. This technique has by now been adapted to stochastic systems [19–21]. The method is asymptotic in time, but at this moment can treat only systems consisting of approximately 50–100 sites. The name DMRG is a bit of a misnomer since no renormalization-group flows are calculated. This may make it hard in some cases to get clear results on issues of universality.

Another approach working within the same spirit was recently introduced by the present authors [22]. In our real-space renormalization-group technique, we apply the so-called standard renormalization method (SRG), also known as the SLAC approach [23], to the quantum Hamiltonian

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associated with the stochastic model. The SLAC approach was introduced in the late 1970s in the study of lattice gauge theories and was subsequently applied to a great number of quantum spin and fermion systems [24]. In a previous paper, we adapted this technique to the study of stochastic systems [22] and applied it to some exactly solvable cases. Surprisingly, in many cases exact results were recovered. We also applied our technique to the contact process, using a small cell of only three sites. The numerical estimates of critical properties that we obtained for this process were within 10% of the best known values. In the present paper, we extend our calculations for the contact process to as large cells as possible. Using underlying properties of the contact process, we are able to get results for cells with up to 37 sites. To the best of our knowledge, this is a ''world record'' for the SRG approach to quantum systems. Combined with good extrapolation techniques, these results allow us to determine very accurate exponents for the contact process. In fact, the results we obtain are of the same accuracy as those obtained from series expansions and simulations, and are more accurate than those coming from the DMRG.

This paper is organized as follows. In the next section, we introduce the contact process and its quantum description. We also translate the duality of the model into a quantum-mechanical language. In Sec. III, we give a brief outline of our real-space renormalization method. In Sec. IV, we show how for the case of the contact process, the RG flow can be calculated from a knowledge of only two matrix elements. In Sec. V, we present the results of our calculations for different system sizes, describe the extrapolation procedures, and compare our results with those in the literature. Finally, we conclude with a discussion in Sec. VI.

II. QUANTUM DESCRIPTION OF THE CONTACT PROCESS

The contact process was originally introduced as a simple model for the spreading of an epidemic [3]. On each site $i(i:1,\ldots,N)$ of a lattice Λ, there is a variable η_i that can take on two values, referred to as A and $\varnothing$. In the contact process, particles A (vacancies $\varnothing$) are interpreted as sick (healthy) individuals. The dynamics of the model is given by a continuous-time Markov chain on the set of all microscopic configurations $\eta \equiv \{\eta_1, \ldots, \eta_N\}$. The following processes are allowed: a sick person can be cured ($A \to \varnothing$) with rate 1 and a healthy individual can become contaminated with a rate $z\lambda/2$, where z is the number of sick neighbors. The conditional probability $P(\eta,t;\eta_0,0)$ that the system is in configuration η at time t given that it was in η_0 at time $t=0$ then obeys the master equation

$$\frac{dP(\eta,t;\eta_0,0)}{dt} = -\sum_{\eta'} H(\eta,\eta')P(\eta',t;\eta_0,0), \qquad (1)$$

where the $2^N \times 2^N$ matrix H, the generator of the Markov chain, depends on the transition rates, i.e., on λ.

The master equation (1) is formally equivalent to a Schrödinger equation in imaginary time. It has therefore become common to introduce a quantum-mechanical notation for a stochastic process. This mapping of a stochastic system onto a quantum-mechanical one is by now quite standard and we will not discuss it here in any detail. We only give a brief review that also fixes the notation that will be used further. To each configuration η, a state vector $|\eta\rangle = \otimes_{i=1}^{N} |\eta_i\rangle$ is associated. The vectors $|\eta_i\rangle$ form the basis vectors of a two-dimensional vector space. It is then natural to use a spin-$\frac{1}{2}$ language to describe this vector space. As usual, a particle (vacancy) will be associated with spin down (up) [26].

Next, we also associate a vector $|P(t)\rangle$ with the conditional probabilities $P(\eta,t;\eta_0,0)$ such that

$$|P(t)\rangle = \sum_{\eta} P(\eta,t;\eta_0,0)|\eta\rangle. \qquad (2)$$

Using this notation, the master equation (1) is rewritten as

$$\frac{d|P(t)\rangle}{dt} = -H|P(t)\rangle. \qquad (3)$$

From now on, we will refer to the matrix H as the Hamiltonian of the stochastic system. For processes with transition rates that involve only nearest-neighbor sites (such as is the case for the contact process), H can be written as a sum of local Hamiltonians $h_{i,i+1}$ that act only on nearest-neighbor sites,

$$H = \sum_i 1_1 \otimes \cdots \otimes 1_{i-1} \otimes h_{i,i+1} \otimes 1_{i+2} \otimes \cdots 1_N. \qquad (4)$$

In the particular case of the contact process, we have (see also [25])

$$h_{i,i+1} = (n_i - s_i^+) \otimes 1_{i+1} + \frac{\lambda}{2}[(v_i - s_i^-) \otimes n_{i+1} + n_i \otimes (v_{i+1} - s_{i+1}^-)], \qquad (5)$$

where the matrices v, n, s^+, and s^- are given by

$$v = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad s^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad s^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \qquad (6)$$

The formal solution of the master equation (3) is

$$|P(t)\rangle = e^{-Ht}|P(0)\rangle.$$

Because of the properties of stochastic matrices, there is always a zero eigenvalue and the real parts of the other eigenvalues of H are never negative. Therefore, asymptotically, for $t \to \infty$ the behavior of $|P(t)\rangle$ is determined by the properties of the ground state(s) of the quantum Hamiltonian. In our RG approach, we study the critical behavior of the stationary state of the stochastic system by applying a real-space renormalization technique originally developed to study ground-state properties of quantum systems.

Expectation values of physical quantities can also easily be rewritten in terms of the quantum notation. With each physical quantity $\mathcal{F}$ (such as the density of particles, correlation functions, etc.), we can associate a quantum-mechanical operator F [with matrix elements $\langle \eta|F|\eta'\rangle = \mathcal{F}(\eta)\delta_{\eta\eta'}$] such that the expectation value of $\mathcal{F}$,

$$\langle \mathcal{F}\rangle(t) = \sum_{\eta} \mathcal{F}(\eta) P(\eta,t;\eta_0,0),$$

can be rewritten as

$$\langle \mathcal{F}\rangle(t) = \langle s|F|P(t)\rangle = \langle s|Fe^{-Ht}|P(0)\rangle,$$

where we have introduced the shorthand notation

$$\langle s| = \sum_{\eta} \langle \eta|. \tag{7}$$

The contact process has a property known as duality [27]. This notion was first introduced in the probabilistic study of interacting particle systems and should not be confused with the concept of duality from equilibrium statistical mechanics. Before proceeding with the renormalization-group study of the contact process, we show how this duality can be derived in the quantum-mechanical language. This formulation of duality was first given in [28]. We will describe it here in a slightly different way, which will be appropriate for the use we want to make of it in our renormalization calculations (see Sec. IV).

We begin by introducing the 2×2 matrix d,

$$d = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and $d_k = 1_1 \otimes 1_2 \cdots \otimes d \otimes \cdots \otimes 1_N$, $\forall k$ (with d on the kth site). It is then easy to check the fundamental property

$$h_k^T = (d_k \otimes d_{k+1}) h_k (d_k \otimes d_{k+1})^{-1},$$

where T stands for transposed. Furthermore, one has $d_k n_k d_k^{-1} = v_k - s_k^+$. From these relations and defining $D = \otimes_k d_k$, one can obtain a useful expression for the expectation value of n_k that gives the density of particles at site k,

$$\begin{aligned}\langle n_k\rangle(t) &= \langle s|n_k e^{-Ht}|P(0)\rangle \\ &= \langle s|D^{-1}Dn_kD^{-1}De^{-Ht}D^{-1}D|P(0)\rangle. \end{aligned}\tag{8}$$

If $|0\rangle$ and $|L\rangle$ denote, respectively, the completely empty and completely full configuration, one has

$$(D^{-1})^T|s\rangle = |0\rangle,$$

$$D|L\rangle = |0\rangle.$$

Therefore, if we take as the initial condition $|P(0)\rangle = |L\rangle$, Eq. (8) becomes

$$\begin{aligned}\langle n_k\rangle(t) &= \langle 0|(v_k - s_k^+)e^{-H^T t}|0\rangle \\ &= \langle 0|e^{-Ht}(v_k - s_k^-)|0\rangle \\ &= 1 - \langle 0|e^{-Ht}|k\rangle. \end{aligned}\tag{9}$$

Here $|k\rangle$ is the state with only one particle, at site k. The matrix element $\langle 0|e^{-Ht}|k\rangle$ appearing in Eq. (9) gives the probability that starting at $t=0$ with one particle at site k, no particles are left in the system at time t. If we introduce the *survival probability* $P_k(t)$ as the probability that if the system is initially in the state $|k\rangle$ there are still particles in the system at time t, we finally get

$$\langle n_k\rangle(t) = P_k(t). \tag{10}$$

This is the duality relation of the contact process. It says that the density of particles at site k when starting from a completely full lattice is the same as $P_k(t)$. For $t\to\infty$, one expects that the steady-state value of the density becomes independent of the initial condition and hence we obtain

$$\langle n_k\rangle_{\rm st} = P_{k,\rm st}. \tag{11}$$

Here, the subscript ''st'' denotes the stationary-state value.

(In the rest of this paper, we will drop the direct product symbol to shorten the notation. We will also drop all unity operators, as their presence is always implicitly assumed.)

III. STANDARD RENORMALIZATION FOR STOCHASTIC SYSTEMS

We briefly review the use of the standard real-space RG for stochastic systems as introduced in a previous paper [22]. For more details, we refer to that work.

As usual in real-space RG approaches, the lattice is divided into cells, each containing b sites. In the case of a one-dimensional system, we can regroup the terms in the Hamiltonian H (4) to write

$$H = \sum_{\alpha} (H_{0,\alpha} + V_{\alpha,\alpha+1}). \tag{12}$$

Here α labels the cells, $H_{0,\alpha}$ contains the intracell terms of H, and $V_{\alpha,\alpha+1}$ denotes the intercell interactions. Next, $H_{0,\alpha}$ is diagonalized exactly. For simplicity, we now only consider the case where the ground state of $H_{0,\alpha}$ is doubly degenerated. We then have two right and two left ground states of $H_{0,\alpha}$ denoted as $|s_1\rangle_\alpha$, $|s_2\rangle_\alpha$ and ${}_\alpha\langle s_1|$, ${}_\alpha\langle s_2|$, which we can normalize as ${}_\alpha\langle s_i|s_j\rangle_\alpha = \delta_{ij}$. We consider one of these states as representing a ''cell vacancy'' state $|\varnothing\rangle_\alpha$ and the other as a ''cell particle'' $|A\rangle_\alpha$ state. These states are used to construct renormalized lattice configurations $|\eta'\rangle = \otimes_\alpha |\eta\rangle_\alpha$, which span a $2^{N/b}$-dimensional subspace $\mathcal{W}$ of the original state space.

The renormalization transformation is now performed by projecting the original Hamiltonian onto $\mathcal{W}$. This is done by means of the matrices

$$T_1=\sum_{\eta'} |e_{\eta'}\rangle\langle \eta'|, \quad T_2=\sum_{\eta'} |\eta'\rangle\langle e_{\eta'}|, \tag{13}$$

where $|e_{\eta'}\rangle$ are the vectors of the standard basis of $\mathcal{W}$. Because of our choice of normalization for the ground states, we get $T_1T_2=1$, the identity operator on $\mathcal{W}$. Finally, T_1 and T_2 are used to calculate the renormalized Hamiltonian H' as

$$H'=T_1HT_2. \tag{14}$$

When the ground state of the intracell parts $H_{0,\alpha}$ is doubly degenerated (as we assumed), it is easy to show that H' is again stochastic.

If we collect the rates appearing in H in a vector $\vec{w}$, Eq. (14) defines a mapping in the parameter space $\vec{w}'=f(\vec{w})$. From this mapping we can determine fixed points, critical exponents, expectation values in the ground state, etc. (One of the rates appearing in the Hamiltonian can always be taken equal to 1. This is merely a fixing of the time scale. The corresponding rate in the renormalized Hamiltonian is not necessarily 1, but we then divide H' by this renormalized rate; the effect of this division is included in the mapping f.)

To fix ideas, let us assume that the equations $\vec{w}'=f(\vec{w})$ have a nontrivial fixed point at $\vec{w}^\star$, with one relevant scaling field (which in a linear approximation is proportional to $\Delta w_1=w_1-w_1^\star$) whose scaling dimension is y_{w_1}. From standard RG theory, it then follows that near criticality the correlation length ξ will diverge as $\xi\sim|\Delta w_1|^{\nu_\perp}$ with

$$\nu_\perp=1/y_{w_1}. \tag{15}$$

In order to determine the order-parameter exponent β, we need first to explain how the particle density in the stationary state $c_{\rm st}(\vec{w})$ can be calculated within our RG scheme. When the system is in the ground state $|s_i\rangle$, and assuming translational invariance, this density is given by

$$c_{\rm st}(\vec{w})=\langle s|n_k|s_i(\vec{w})\rangle, \tag{16}$$

where we have now explicitly indicated the dependence of the ground state on the transition rates $\vec{w}$. Under the renormalization, this expectation value transforms as [22]

$$c_{\rm st}(\vec{w})=a(\vec{w})c_{\rm st}(\vec{w}'). \tag{17}$$

Here we assumed, as will turn out to be the case for the contact process, that the renormalized particle operator $n_k'=T_1n_kT_2$ is proportional to n_k, i.e., $n_k'=a(\vec{w})n_k$. The relation (17) can be iterated along the RG flow, and hence the density of particles can be obtained as an infinite product if one knows the density at the (trivial) fixed point $\vec{w}_t^\star$ that attracts $\vec{w}$,

$$c_{\rm st}(\vec{w})=\left[\prod_{i=0}^{\infty} a(\vec{w}^{(i)})\right]c_{\rm st}(\vec{w}_t^\star). \tag{18}$$

In principle, other expectation values can be calculated in a similar way.

To conclude this section, we show how Eq. (17) can be used to calculate the exponent β. Near $\vec{w}^\star$, we get for the singular part of $c_{\rm st}$,

$$c_{\rm st}(\Delta w_1)=a(\vec{w}^\star)c_{\rm st}(b^{y_{w_1}}\Delta w_1). \tag{19}$$

We write

$$a(\vec{w}^\star)=b^{\beta/\nu_\perp} \tag{20}$$

and get from Eqs. (19) and (15)

$$c_{\rm st}(\Delta w_1)\sim(\Delta w_1)^\beta, \tag{21}$$

which justifies Eq. (20). Hence, β can be obtained from $a(\vec{w}^\star)$.

IV. RENORMALIZATION OF THE CONTACT PROCESS

This section is a more technical one, and shows explicitly how we calculate the RG flow for the contact process.

We start by dividing the lattice in blocks of length b. The terms of the Hamiltonian (4) and (5) are regrouped. We therefore introduce the following short-hand notations:

$$h_i^1=n_i-s_i^+, \tag{22}$$

$$h_i^2=\frac{\lambda}{2}[(v_i-s_i^-)n_{i+1}+n_i(v_{i+1}-s_{i+1}^-)], \tag{23}$$

which are, respectively, the generators of the processes $A\to\varnothing$ and $A+\varnothing\to A+A, \varnothing+A\to A+A$. Notice that each of these terms itself has the property of duality.

It is now important to remark that the regrouping in intracell and intercell parts of the Hamiltonian is not unique. A natural attempt is to take into $H_{0,\alpha}$ all the terms that act on the sites inside the cell, hence take the intracell Hamiltonian as that of a contact process for a system of b sites with open boundary condition: $H_{0,\alpha}=\Sigma_{i=1}^b h_{\alpha,i}^1+\Sigma_i^{b-1}h_{\alpha,i}^2$ (here the first subindex labels the cell, while the second indicates the site in the given cell). This choice, however, is not suitable for us since in this case the intracell Hamiltonian has only one ground state, which is the trivial empty lattice $|s_1\rangle_\alpha=\otimes_{i=1}^b|\varnothing\rangle_{\alpha,i}$.

As we argued in the preceding section, $H_{0,\alpha}$ should have two ground states, one representing the effective vacancy of cell α and one representing the effective particle. It is the second one that is missing. To solve this problem, we ''force'' $H_{0,\alpha}$ to have an active ground state by removing h_i^1 on the central site of the cell. This resembles the so-called self-dual renormalization group introduced earlier in the study of quantum models such as the Ising model in a transverse field [29].

From now on, we choose b odd, $b=2n-1$, and take

$$H_{0,\alpha}=\sum_{i=1}^{n-1} h^1_{\alpha,i}+\sum_{i=n+1}^{b} h^1_{\alpha,i}+\sum_{i=1}^{b-1} h^2_{\alpha,i}. \tag{24}$$

The operator $h^1_{\alpha,n}$ that we removed from $H_{0,\alpha}$ is then of course added to $V_{\alpha,\alpha+1}$ concluding the regrouping of the terms occurring in the Hamiltonian.

The hardest part of the RG is the calculation of the two right and left ground states of $H_{0,\alpha}$ for b as large as possible. Analytically this can only be done for b up to five sites. To make a reliable extrapolation for $b\to\infty$ possible, we need a good numerical algorithm to get to larger b. Before we turn to this point in the next section, there are a few analytical considerations that can be made and that will turn out to be extremely useful in studying large cell sizes.

We start with the two right ground states $|s_1\rangle_\alpha$, $|s_2\rangle_\alpha$ of $H_{0,\alpha}$ as defined in Eq. (24). The first one is again trivial $|s_1\rangle_\alpha=\otimes^b_{i=1}|\varnothing\rangle_{\alpha,i}$. To get an idea of how the active ground state $|s_2\rangle_\alpha$ looks, we rewrite Eq. (24) as

$$H_{0,\alpha}=H^l_\alpha+H^r_\alpha, \tag{25}$$

where

$$\begin{aligned} H^r_\alpha&=\sum_{i=1}^{n-1} h^1_{\alpha,i}+\sum_{i=1}^{n-1} h^2_{\alpha,i},\\ H^l_\alpha&=\sum_{i=n+1}^{b} h^1_{\alpha,i}+\sum_{i=n}^{b-1} h^2_{\alpha,i}. \end{aligned} \tag{26}$$

Physically H^r_α (H^l_α) is the stochastic generator of the contact process on a lattice of n sites without the process $A\to\varnothing$ on the right (left) site. These operators have again a trivial right ground state (the empty lattice) and a nontrivial (active) right ground state. To get a better grasp on the latter one, we first notice that H^r_α has no ($A\to\varnothing$) term on site (α,n). Since this is the only reaction destroying A's on that site, there are no transitions possible from configurations with an A on site (α,n) to configurations with no particle at (α,n). Denote the subspace spanned by the former configurations by $\mathcal{V}$. Then, H^r_α defines a stochastic process on $\mathcal{V}$, implying that H^r_α must have a ground state in this subspace. This ground state clearly cannot be the empty lattice since this is not an element of $\mathcal{V}$. We therefore conclude that this state is the active ground state of H^r_α and that it has with probability 1 a particle A at site n. The same can be said for H^l_α. As a consequence, we can write the active ground state as

$$\begin{aligned} &|\psi\rangle^r_\alpha\otimes|A\rangle_{\alpha,n} \quad \text{for } H^r_\alpha,\\ &|A\rangle_{\alpha,n}\otimes|\psi\rangle^l_\alpha \quad \text{for } H^l_\alpha, \end{aligned} \tag{27}$$

where $|\psi\rangle^r_\alpha$ and $|\psi\rangle^l_\alpha$ are states on a lattice with $n-1$ sites (to get some idea of the state $|A\rangle_{\alpha,n}\otimes|\psi\rangle^l_\alpha$, we show in Fig. 1 the behavior of the particle density in this state for a system of eight sites).

It then follows from Eq. (26) that $|\psi\rangle^r_\alpha\otimes|A\rangle_{\alpha,n}\otimes|\psi\rangle^l_\alpha$ is the active ground state of $H_{0,\alpha}$. Hence, if we can find the right ground state of the ''contact process'' determined by H^r_n on a lattice of n sites, we can construct the ground state of the process defined by $H_{0,\alpha}$ on a lattice of $b=2n-1$ sites (since $|\psi\rangle^l_\alpha$ can easily be obtained by reflection once $|\psi\rangle^r_\alpha$ is known). In conclusion, we have

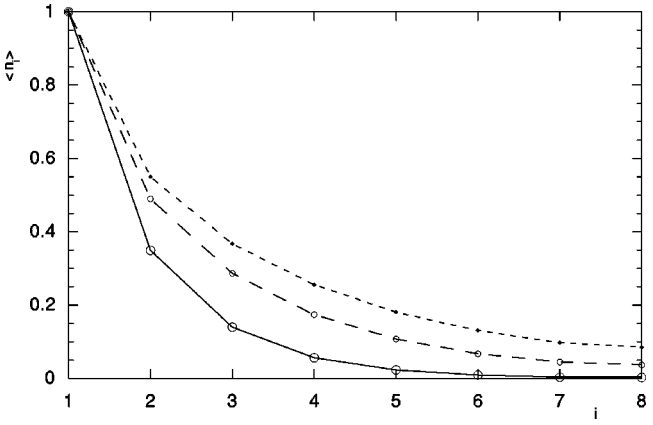

FIG. 1. Average density of particles $\langle n_i\rangle$ in the state $|A\rangle_{\alpha,n}\otimes|\psi\rangle^l_\alpha$ for $n=8$, at different values of λ. The curves are (top to bottom) at $\lambda=4,3.298(\lambda_c)$ and $\lambda=2$.

$$\begin{aligned} |s_1\rangle_\alpha&=\otimes^b_{i=1}|\varnothing\rangle_{\alpha,i},\\ |s_2\rangle_\alpha&=|\psi\rangle^r_\alpha\otimes|A\rangle_{\alpha,n}\otimes|\psi\rangle^l_\alpha. \end{aligned} \tag{28}$$

For the left ground states of $H_{0,\alpha}$, we always have the trivial ground state ${}_\alpha\langle s|=\Sigma_\eta\langle\eta|$ and a nontrivial one. To find the latter, we exploit the duality of $H_{0,\alpha}$. Indeed, from this duality $H^T_{0,\alpha}=BH_{0,\alpha}B^{-1}$, it follows that for any right ground state $|s_k\rangle_\alpha$ of $H_{0,\alpha}$, $(B|s_k\rangle_\alpha)^T$ is a left ground state. Denoting ${}^{r,l}_\alpha\langle\phi|=(B|\psi\rangle^{r,l}_\alpha)^T$, we have

$$\begin{aligned} {}_\alpha\langle s_1|&={}^r_\alpha\langle\phi|\otimes\langle\varnothing|\otimes{}^l_\alpha\langle\phi|,\\ {}_\alpha\langle s_2|&={}_\alpha\langle s|-{}_\alpha\langle s_1|. \end{aligned} \tag{29}$$

We take ${}_\alpha\langle s_2|$ of this form because of normalization reasons. Our choice guarantees that ${}_\alpha\langle s_i|s_j\rangle_\alpha=\delta_{i,j}$, which we need to conserve stochasticity as explained in the preceding section.

To conclude the calculation, we perform the renormalization transformation (14) on H. Without any further information on $|\psi\rangle^{r,l}_\alpha$, we know that the ground states (28) and (29) of $H_{0,\alpha}$ are properly normalized and have left-right symmetry, and moreover we know the state of the central site. Using these three properties, it is straightforward to show that the renormalized Hamiltonian H' contains the same terms as H, there is therefore no proliferation of interactions, and that the RG equation for the rate λ is of the form

$$\lambda'=\lambda\frac{v^2}{w^2}, \tag{30}$$

where

$$v=\langle s|n_{\alpha,1}(|\psi\rangle^r_\alpha\otimes|A\rangle_{\alpha,n}),$$

TABLE I. Critical parameters as calculated by the RG method for a block of b sites, together with the results coming from other approaches.

b	λ_c	$\nu_\perp$	$\beta/\nu_\perp$
9	3.228740192229	1.100222670443	0.300770659640
11	3.232841532095	1.099704726572	0.291239313449
13	3.236622341324	1.099306840428	0.284829626291
15	3.240001893307	1.098993499409	0.280220582724
17	3.243002363779	1.098741258486	0.276745857289
19	3.245669884031	1.098534370274	0.274032449042
21	3.248051604747	1.098361971472	0.271855062319
23	3.250189366461	1.098216363626	0.270069478446
25	3.252118590397	1.098091954504	0.268579043181
27	3.253868772889	1.097984591005	0.267316523888
29	3.255464377754	1.097891127814	0.266233690895
31	3.256925727824	1.097809141091	0.265295037853
33	3.258269778268	1.097736734033	0.264473840447
35	3.259510752614	1.097672401665	0.263749596182
37	3.260660654464	1.097614935125	0.263106311367
$b\to\infty$	3.2982(2)	1.09682(2)	0.2534(4)
series expansion [15,14]	3.29785(2)	1.0969(1)	0.2520(1)
diagonalization [34]	3.29792(2)	1.09681(1)	0.256(1)
simulations [33]		1.09684(1)	0.25208(1)
series expansions [39]		1.096854(4)	0.252072(11)
DMRG [20]		1.08(2)	0.249(3)

$$
\begin{aligned}
w &= ({}^r_\alpha\langle\phi|\otimes_{\alpha,n}\langle\varnothing|)s_n^+(|\psi\rangle^r_\alpha\otimes|A\rangle_{\alpha,n}) \\
&= {}^r_\alpha\langle\phi|\psi\rangle^r_\alpha \\
&= {}^r_\alpha\langle\psi|B^T|\psi\rangle^r_\alpha .
\end{aligned}
\tag{31}
$$

This means we can generate the RG map for the contact process with cell length $b=2n-1$ by calculating two matrix elements in the right ground state of the contact process H^r_α on a lattice of only n sites. In this way, it is possible to perform the RG for rather large cell sizes. Moreover, for each cell size, the calculations that have to be performed are rather limited.

A special feature of our renormalization-group calculation is the absence of proliferation of interactions. It is not very clear to us what the precise conditions are on the effective cell states that guarantee this simplification. Certainly this feature would disappear if we would extend our calculations to higher order along the lines discussed for quantum spin systems in [35].

V. RESULTS

We applied the RG procedure described in the preceding sections to the contact process for block sizes $b=3,5,\ldots$. For each size $b=2n-1$ we had to calculate the nontrivial ground state of the nonsymmetrical $2^n\times 2^n$ matrix H^r_α. Analytically, this was only possible for $b=3$ and $b=5$. For larger block sizes, we turned to numerical diagonalization methods, in particular the Arnoldi algorithm [30]. This algorithm is designed to calculate eigenvalues and eigenvectors of an extreme part of the spectrum (in our case the low-lying part) of large nonsymmetrical matrices. When the algorithm converges, it produces very precise estimates. Since for stochastic systems we know the value of the ground-state energy exactly, we have a reliable criterion to decide on convergence and hence a very powerful diagonalization tool. Using this method, we were able to perform the RG calculations up to $b=37$.

For each b value, the location of the critical point λ_c and the critical exponent $\nu_\perp$ were calculated using the methods explained in Secs. III and IV. In order to determine also the exponent ratio $\beta/\nu_\perp$, we need to calculate the quantity $a(\lambda)$ at the critical point [see Eq. (20)]. This requires the calculation of some extra matrix elements. Our results for λ_c and the two critical exponents are given in Table I.

We have extrapolated the results of the RG calculations using the BST algorithm [31], which is known to be a good tool to extrapolate finite lattice data [32]. The results are also included in Table I.

In Table I we also compare our extrapolations with those that can be found in the literature and are based on a variety of other techniques. The results in the second row were obtained from a numerical diagonalization of the Hamiltonian for the contact process on a finite lattice. In that case, the exponent β was not calculated, but we used a scaling relation [1] to obtain this exponent from estimates of $\nu_\perp$ and the

exponent δ. The estimates in the three last rows are not for the contact process itself, but for other models in the same universality class.

As can be seen, the results of our RG technique compare very well with those of the other techniques, especially for the location of the critical point and the correlation length exponent. The value of $\beta/\nu_\perp$ is somewhat less precise. However, in comparison with the standard of real-space renormalization calculations, the current results must be considered to be extremely precise.

VI. CONCLUSIONS

In this paper, we have applied a real-space renormalization-group technique, originally developed for quantum systems, to the contact process. Using some analytical properties, such as the duality of the process, we have been able to carry out the renormalization for rather large cell sizes. Together with a suitable extrapolation, this has yielded estimates for critical properties that are of very high accuracy.

In our previous paper [22], we applied the technique to simple reaction-diffusion processes that do not have a phase transition. There we showed that in some cases our RG technique was able to reproduce exact results. Combining the results of the two papers, we believe that it is fair to say that the technique is able to give accurate results for the stationary state of stochastic systems with one type of particle (or stated otherwise, in which the variable at each site can be in two different states), when only nearest-neighbor interactions are involved. Of course, it may be so that for any particular model some ''cooking'' is necessary in order to obtain good results. But that is a quite general limitation of real-space renormalization approaches.

At this moment, there are two obvious directions in which to develop this RG method. First, one may consider processes in which more than two particles are involved. One can think, for example, of the process $\varnothing+A+\varnothing\rightarrow A+A+A$ that appears in the branching and annihilating walks with an even offspring, a model that belongs to the PC university class. In that case, the Hamiltonian of the process contains three site interactions. As argued in our previous paper, it is necessary to extend the current RG procedure to higher order to be able to obtain a renormalized Hamiltonian with three site interactions (in [35] such a higher-order extension of the standard RG method is discussed in the context of quantum spin chains). This higher-order extension requires knowledge of all the eigenvalues and eigenvectors of the cell Hamiltonian, which severely restricts the cell sizes that can be studied, since algorithms such as the Arnoldi or Lanczos procedures are only able to give good estimates of low-lying eigenvectors and eigenvalues. Moreover, in stochastic systems, eigenvalues can be complex, which in turn can give rise to parameter flows that are complex. We have tried this kind of higher-order technique in a preliminary study of a nonequilibrium Ising model [36] whose transition is believed to be in the PC class, and which also has a duality [37]. In that calculation, we encountered this problem of complex eigenvalues, and at this moment it is unclear to us how to proceed in this direction.

A more promising approach extends the techniques introduced here to models with several types of particles, or with more than two states per site. If one restricts again the interactions to be of nearest-neighbor type, there are no fundamental problems to apply our RG technique. Several of the interesting processes mentioned in the Introduction belong to this class of models. One may think of branching and annihilating walks with two types of particles and exclusion, the model originally studied by Van Wijland *et al.* [38], etc.

Another very interesting model of this latter type was recently introduced by Hinrichsen [9]. It is a generalization of the contact process in which at each site there can be n ''empty'' or nonactive states. For $n=1$, the model coincides with the contact process studied here. For $n=2$, the generalized contact process is believed to be in the PC universality class, whereas for $n>2$ the critical behavior has not been determined yet [9]. Because this model is a natural extension of the contact process, and given the success of our RG method for that model, we believe it is an example of an interesting model that could be studied successfully with our approach. Unfortunately, the model has no obvious duality. Moreover, since at each site the system can be in $n+1$ states, the calculations will by necessity be restricted to smaller b values. Nevertheless, for $n=2$ it should still be possible to reach cell sizes $b\approx 20-25$. In this way, we hope it will be possible to obtain rather accurate exponent estimates for the PC universality class. We plan to present results of an RG study of the $n=2$ Hinrichsen model in a forthcoming paper.

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